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ON COMBINATORY ALGEBRAS AND THEIR EXPANSIONS*

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Introduction

Combinatory algebras (defined below) provide a useful framework for studying functions and functional application since algebraically definable functions over the combinatory algebra are representable (combinatory completeness). Unfortunately, these structures are not in general rich enough to provide a sound interpretation for the untyped λ-calculus. In particular, rule $(\xi): u = v \vdash \lambda x.u = \lambda x.v$, is not valid in all combinatory algebras. Thus stronger structures, called λ-models, are needed to intepret the λ-calculus.

Since the original inverse limit construction by Scott, several λ-models have been defined. In particular, Scott [13] has also defined another model, $(P\omega, \cdot, \Psi)$, over the set of all subsets of the natural numbers. Scott's ideas have been developed in several directions (see [8, 1, 7, 4, 11, 2, 3] for applications to constructive set theory, recursion theory in higher types, and for model-theoretic results). In Stoy [14] there is a rather detailed presentation of the main area of application of the semantics of λ-calculus to theoretical computer science: the denotational semantics of programming languages.

While not all combinatory algebras can be made into λ-models, many can. In this paper we examine such expansions of combinatory algebras. Most of the focus here is on Scott's $P\omega$ model, where we show that there is a unique expansion of the applicative structure $\langle P\omega, \cdot\rangle$ to a λ-model. We also show the uniqueness of K

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and S in this model under further stability conditions. As a corollary we obtain the result that $\langle P\omega, \cdot\rangle$ is not isomorphic to Plotkin–Engeler's $\langle D_A, \cdot\rangle$ for any A. Finally we present a general result on the expansion of interiors of a class of λ-models.

In [11], a detailed analysis of the theories and structures of variants of the $\langle D_A, \cdot\rangle$ models is carried out. Using the results of this analysis and an indirect argument, a different proof was given of the uniqueness of the expansion of $\langle P\omega, \cdot\rangle$. The proof contained in this paper is much more elementary and algebraic in nature.

1. Definitions

Combinatory algebras are given by an applicative structure, $\langle D, \cdot\rangle$, and a k and s in D which interpret the K and S of combinatory logic. Thus combinatory algebras have a straightforward algebraic (first-order) characterization: $\exists K, S, K \neq S$ such that

(K) $\quad \forall x \forall y\ (Kxy = x)$,

(S) $\quad \forall x \forall y \forall z\ (Sxyz = xz(yz))$.

The definition of λ-models is somewhat more complicated. One form of the definition is given as follows. Let $\langle D, \cdot\rangle$ be an applicative structure. Let

$$(D^n \to D) = \{f : D^n \to D \mid \exists d \in D \forall \boldsymbol{e} \in D^n, f(\boldsymbol{e}) = d \cdot \boldsymbol{e}\}$$

(the set of representable functions over D^n). A λ-*model* is a $\langle D, \cdot, \Psi\rangle$ such that

(1) $\Psi : (D \to D) \to D$ and $\Psi(f) \cdot e = f(e)$ for all $e \in D$,

(2) for all $f \in (D^{n+1} \to D)$, $\lambda \boldsymbol{e} \in D^n.\Psi(\lambda d \in D.f(d, \boldsymbol{e}))$ is in $(D^n \to D)$ (i.e., if f is representable, then so is the map $\boldsymbol{e} \mapsto \Psi(\lambda d \in D.f(d, \boldsymbol{e}))$).

The interest in λ-models arises from the fact that a λ-model $\langle D, \cdot, \Psi\rangle$ provides an interpretation of λ-terms via the following definition:

$$\|x\|\sigma = \sigma(x),$$

$$\|(MN)\|\sigma = (\|M\|\sigma)\cdot(\|N\|\sigma),$$

$$\|\lambda x M\|\sigma = \Psi(f) \quad \text{where } f(e) = \|M\|\sigma_x^e.$$

In the above, σ is a mapping from the set of variables to D and σ_x^e is defined so that, for $y \neq x$, $\sigma_x^e(y) = \sigma(y)$ and $\sigma_x^e(x) = e$.

It is easy to show that $\|\cdot\|\sigma$ is well defined, in particular that $f \in (D \to D)$. Moreover, this interpretation clearly satisfies (β) $(\lambda x.M)N = [N/x]M$ and the other axioms of λ-calculus over the λ-model. It is not immediately apparent from the above definitions that each λ-model is a combinatory algebra as well. Meyer [12] has given an elegant algebraic characterization of λ-models that makes this connection much clearer. A *combinatory model* is a triple $\langle D, \cdot, \rangle$ such that $\langle D, \cdot\rangle$ is a combinatory algebra, and $\varepsilon \in D$ satisfies:

$(\varepsilon 1)$ $\quad \forall x \forall y\ (\varepsilon \cdot x \cdot y = x \cdot y)$,

$(\varepsilon 2)$ $\quad \forall x \forall y\ (\forall z (x \cdot z = y \cdot z) \to \varepsilon \cdot x = \varepsilon \cdot y)$.

Thus ε picks out a canonical element from each collection of elements of D which have the same functionality. We say a combinatory model is *stable* if $\varepsilon \cdot \varepsilon = \varepsilon$.

How do ε and λ-abstraction (equivalently Ψ) relate? Given a Combinatory Algebra, $\langle D, \cdot \rangle$, is there a unique way of choosing ε (or Ψ)? Are K and S also uniquely determined? Meyer answered the first of these questions. In Section 2 we examine the other two.

Proposition 1.1. (i) The class of stable combinatory models coincides with the class of λ-*models. More precisely, there is a natural one-to-one correspondence between stable combinatory models and* λ-*models.*

(ii) *A combinatory model* $\langle D, \cdot, \varepsilon \rangle$ *is stable iff* $\varepsilon = \|\lambda xy \cdot xy\|\sigma$.

Proof. (i) This is easy. Let us be given a λ-model $\langle D, \cdot, \Psi \rangle$, set $\varepsilon = \Psi(\lambda e \in D. \Psi(\lambda d \in D. e \cdot d))$. Conversely, given a stable combinatory model $\langle D, \cdot, \varepsilon \rangle$ set $\Psi(f) = \varepsilon \cdot d$ if d represents f. (For details of the verification, see [12] or [10].)

$$\lambda\beta \vdash (\lambda xy.xy)(\lambda xy.xy) = \lambda xy.xy.$$

$$(\Rightarrow) \quad \|\lambda xy.xy\|\sigma = \Psi(\lambda e \in D. \Psi(\lambda d \in D. e \cdot d)) = \Psi(\lambda e \in D. \varepsilon \cdot e) = \varepsilon \cdot \varepsilon = \varepsilon$$

by the definition of Ψ given in the proof of part (i) and by stability.

For further discussion of λ-models, see [12, 8, 5, 9].

2 $P\omega$ and abstraction

In this section we investigate Scott's $P\omega$ model for λ-calculus. ($P\omega$ is the set of all subsets of ω.) In order to form a combinatory algebra or λ-model from $P\omega$ we must first define an interpretation for formal application. Let $\{e_n\}_{n \in \omega}$ and $\langle \cdot, \cdot \rangle$ be effective codings of finite sets and of pairs, say as given in [15]. Let $(a_1, a_2, \ldots, a_n)$ abbreviate $\langle a_1, \langle a_2, \langle \ldots, a_n \rangle \cdots \rangle\rangle\rangle$. Then, for $d, e \in P\omega$,

$$d \cdot e = \{m \mid \exists e_n \subseteq e, (n, m) \in d\}.$$

This is the Myhill–Shepherdson–Rogers notion of application given for the definition of enumeration operators (see [15]), and is monotonic in both arguments. With this definition of application it is easy to see that $\langle P\omega, \cdot \rangle$ is a combinatory algebra (take $k = \{(m, n, p) \mid p \in e_m\}$ and $s = \{(m, n, p, q) \mid q \in (e_m \cdot e_p) \cdot (e_n \cdot e_p)\}$).

In a λ-model, Ψ (or ε) provides an interpretation of functional abstraction (similar to LISP's LAMBDA). In $P\omega$, we say f is *continuous* iff $f(d) = \bigcup\{f(e_n): e_n \subseteq d\}$. Thus, a continuous function is determined by its action on finite subsets of ω. For any continuous f, set $\Psi(f) = \{(n, m) \mid m \in f(e_n)\}$. Clearly $\Psi(f) \cdot d = f(d)$, thus verifying (1) of the definition of λ-models. It is routine to verify (2) of the definition, and hence $\langle P\omega, \cdot, \Psi\rangle$ is a λ-model. One can also easily verify that $\langle P\omega, \cdot, \varepsilon\rangle$ is a combinatory model if $\varepsilon = \{(m, n, p) \mid p \in e_m \cdot e_n\}$. Note that the λ-model $\langle P\omega, \cdot, \Psi\rangle$ and the combinatory model $\langle P\omega, \cdot, \varepsilon\rangle$ for Ψ and ε as defined above are related by the natural one-to-one correspondence given in Proposition 1.1(i).

Stoy [14] mainly uses Scott's $P\omega$ model in his development of denotational semantics. The following theorem shows that in $P\omega$ there is a unique ε and hence Ψ.

Theorem 2.1. *Let $\langle P\omega, \cdot\rangle$ be the Myhill–Shepherdson applicative structure. Then there is a unique ε such that $\langle P\omega, \cdot, \varepsilon\rangle$ is a stable combinatory model (and this gives Scott's λ-model).*

Proof. If $\varepsilon = \{(m, n, p) \mid p \in e_m \cdot e_n\}$ as given previously, then it is easy to verify that $\langle P\omega, \cdot, \varepsilon\rangle$ is a combinatory model (use the k and s given before). The proof that this ε is unique will proceed by a series of lemmas. Let

$$\varepsilon_0 = \{(m, n, p) \mid e_m = \{(n, p)\}\} \quad \text{and} \quad \varepsilon_1 = \{(m, n, p) \mid \exists e_q \subseteq e_n, e_m = \{(q, p)\}\}.$$

Suppose $\langle P\omega, \cdot, \varepsilon'\rangle$ is a stable combinatory model. We will show $\varepsilon' = \varepsilon$. The proof will proceed by the following lemmas.

Lemma 2.2. *If ε' satisfies axiom (ε1), then $\varepsilon_0 \subseteq \varepsilon' \subseteq \varepsilon$.*

Lemma 2.3. *If ε' satisfies axioms (ε1) and (ε2), then $\varepsilon_1 \subseteq \varepsilon'$.*

Lemma 2.4. *If $\varepsilon_1 \subseteq \varepsilon' \subseteq \varepsilon$, then $\varepsilon' \cdot \varepsilon' = \varepsilon$.*

We postpone the proof of the lemmas to Appendix A and finish the proof of the theorem.

Proof of Theorem 2.1 (*continued*). If $\langle P\omega, \cdot, \varepsilon'\rangle$ is a stable combinatory model, then, by Lemmas 2.2 and 2.3, $\varepsilon_1 \subseteq \varepsilon' \subseteq \varepsilon$. Thus by stability and Lemma 2.4, $\varepsilon' = \varepsilon' \cdot \varepsilon' = \varepsilon$. Thus $\varepsilon' = \varepsilon$.

Engeler [6] has shown how to construct a λ-model from a non-empty set A. (Plotkin, in 1972, had given a similar notion of set-theoretic application.) We write this model as $\langle D_A, \cdot, \Psi\rangle$. The equational theories of $\langle D_A, \cdot\rangle$ and $\langle P\omega, \cdot\rangle$ are the same and in fact each may be isomorphically embedded into the other (see [10]). However, in contrast to the above theorem, Longo [10] has shown that there are several possible choices of ε which make $\langle D_A, \cdot, \varepsilon\rangle$ into a combinatory model. In fact, if $|A|$ is the cardinality of A, there are $2^{|A|}$ possible choices. Some of them even satisfy different sets of equations between λ-terms. Thus we obtain the following.

Corollary 2.5. *$\langle P\omega, \cdot\rangle$ and $\langle D_A, \cdot\rangle$ are not isomorphic for any A.*

Proof. The proof follows by Theorem 2.1 and Longo's result.

If one drops the condition on stability, $\langle P\omega, \cdot\rangle$ yields more than one combinatory model, $\langle P\omega, \cdot, \varepsilon'\rangle$. These others also provide models for the λ-calculus (see Section 1).

Theorem 2.6. *For all $\varepsilon' \in P\omega$, $\langle P\omega, \cdot, \varepsilon'\rangle$ is a combinatory model if and only if $\varepsilon_1 \subseteq \varepsilon' \subseteq \varepsilon$.*

Proof. ($\Rightarrow$) Follows from Lemmas 2.2 and 2.3.

($\Leftarrow$) Let

$$k = \{(m, n, p) \mid p \in e_m\} \quad \text{and} \quad s = \{(m, n, p, q) \mid q \in e_m \cdot e_p \cdot (e_n \cdot e_p)\}.$$

As before one can easily see that $\langle P\omega, \cdot\rangle$ is a combinatory algebra using this k and s. Also for $\varepsilon_1 \subseteq \varepsilon' \subseteq \varepsilon$ and for all $d \in P\omega$, $\varepsilon' \cdot d = \varepsilon \cdot d$ (see Lemma 2.7 below). Thus ε' satisfies (ε1) and (ε2) since ε does.

Lemma 2.7. *Let $\langle P\omega, \cdot, \varepsilon\rangle$ be a stable combinatory model. If $\langle P\omega, \cdot, \varepsilon'\rangle$ is a combinatory model, then*

$$\forall d, \quad \varepsilon \cdot d = \varepsilon' \cdot d.$$

Proof. By Lemmas 2.2 and 2.3, $\varepsilon_1 \subseteq \varepsilon' \subseteq \varepsilon$. By an easy computation, one can verify that, for all d, $\varepsilon_1 \cdot d = \varepsilon \cdot d$. Thus, by monotonicity, $\varepsilon' \cdot d = \varepsilon \cdot d$.

The difficulty with using an arbitrary ε' with $\varepsilon_1 \subseteq \varepsilon' \subseteq \varepsilon$ is that now K, S and ε' may be unrelated. We say $\langle D, \cdot, K, S(,\varepsilon)\rangle$ is a KS-expansion of a combinatory algebra (model) $\langle D, \cdot (,\varepsilon)\rangle$ if S and K satisfy the axioms (S) and (K). Let $\langle D, \cdot, K, S, \varepsilon\rangle$ be a KS-expansion of a combinatory model and let $b = \|\lambda xyz.x(yz)\|\sigma$, the composition operator. Set

$$(\varepsilon 2, 3) \quad \varepsilon_2 = b \cdot \varepsilon \cdot (b \cdot \varepsilon) \quad \text{and} \quad \varepsilon_3 = b \cdot \varepsilon \cdot (b \cdot \varepsilon_2).$$

Proposition 2.8. *Let $\langle D, \cdot, K, S, \varepsilon\rangle$ be a KS-expansion of a stable combinatory model. Then*

(i) $\|\lambda xy.x\|\sigma = \varepsilon_2 \cdot K$,

(ii) $\|\lambda xyz.xz(yz)\|\sigma = \varepsilon_3 \cdot S$.

Proof. For the proof, see Meyer [12].

This relationship leads to the following characterization of 'nice' K and S.

Proposition 2.9. *Let $\langle D, \cdot, K, S, \varepsilon\rangle$ be a KS-expansion of a stable combinatory model. Then*

(i) $K = \varepsilon_2 \cdot K$ iff $K = \|\lambda xy.x\|\sigma$,

(ii) $S = \varepsilon_3 \cdot S$ iff $S = \|\lambda xyz.xz(yz)\|\sigma$.

Proof. Trivial by Proposition 2.8.

We say that a KS-expansion of a combinatory model, $\langle D, \cdot, K, S, \varepsilon\rangle$ is KS-stable if $K = \varepsilon_2 \cdot K$ and $S = \varepsilon_3 \cdot S$. It is easy to check that in any such model $\varepsilon' = b(SKC)$ satisfies $\varepsilon' \cdot \varepsilon' = \varepsilon'$. Moreover, this ε' need not equal the ε of the expanded combinatory model. Notice, however, that any λ-model naturally yields a KS-stable combinatory model: just choose K and S as in the right-hand side of Proposition 2.9(i), (ii). In particular, for the stable $\langle P\omega, \cdot, \varepsilon\rangle$ as in Theorem 2.1 there are a natural K_ε and S_ε making the expanded combinatory model, $\langle P_\omega, \cdot, K_\varepsilon, S_\varepsilon, \varepsilon\rangle$, KS-stable as well as stable.

The following technical result will be used to prove that K_ε and S_ε are the unique K and S making $\langle P\omega, \cdot, K, S, \varepsilon\rangle$ KS-stable.

Lemma 2.10. *Let $\langle P\omega, \cdot, \varepsilon\rangle$ be as in Theorem 2.1. Let ε_2 and ε_3 be as in (ε2, 3). Assume now that $\langle P\omega, \cdot, \varepsilon'\rangle$ is a combinatory model and let ε'_2 and ε'_3 be as in (ε2, 3) with respect to ε'. Then*

(i) $\forall e, e' (e' = \varepsilon'_2 e'$ *and* $e = \varepsilon_2 \cdot e' \Rightarrow e = e')$,

(ii) $\forall e, e' (e' = \varepsilon'_3 e'$ *and* $e = \varepsilon_3 \cdot e' \Rightarrow e = e')$.

Proof. (i) Suppose $e' = \varepsilon'_2 \cdot e'$ and $e = \varepsilon_2 \cdot e'$. Just compute

$$\begin{aligned} e' = \varepsilon'_2 \cdot e' &= b' \cdot \varepsilon' \cdot (b' \cdot \varepsilon')e' \\ &= \varepsilon' \cdot (b' \cdot \varepsilon' \cdot e') \quad \text{by the definition of } b' \\ &= \varepsilon \cdot (b' \cdot \varepsilon' \cdot e') \quad \text{by Lemma 2.7.} \end{aligned}$$

Now, for all d,

$$\begin{aligned} b' \cdot \varepsilon' \cdot e' \cdot d = \varepsilon' \cdot (e' \cdot d) &= \varepsilon \cdot (e' \cdot d) \quad \text{again by Lemma 2.7,} \\ &= b \cdot \varepsilon \cdot e' \cdot d \quad \text{by the definition of } b. \end{aligned}$$

Then $\varepsilon \cdot (b' \cdot \varepsilon' \cdot e') = \varepsilon \cdot (b \cdot \varepsilon \cdot e')$ by (ε2). Finally,

$$e' = \varepsilon(b' \cdot \varepsilon' \cdot e') = \varepsilon(b \cdot \varepsilon \cdot e') = b \cdot \varepsilon \cdot (b \cdot \varepsilon) \cdot e' = \varepsilon_2 \cdot e' = e$$

by assumption. The proof of (ii) is similar.

We now show K_ε and S_ε are unique.

Theorem 2.11. *Let $\langle P\omega, \cdot, \varepsilon\rangle$ be as in Theorem 2.1. Assume that $\langle P\omega, \cdot, K', S', \varepsilon'\rangle$ is a KS-stable combinatory model. Then $K' = K_\varepsilon$ and $S' = S_\varepsilon$.*

Proof. Since $\langle P\omega, \cdot, K', S', \varepsilon' \rangle$ is KS-stable, $\varepsilon_2' \cdot K' = K'$ and $\varepsilon_3 \cdot S' = S'$.

Also since $\langle P\omega, \cdot, K', S', \varepsilon \rangle$ is an expanded stable combinatory model (note that no interaction between K', S', and ε is required), Proposition 2.8 implies $K_\varepsilon = \varepsilon_2 \cdot K'$ and $S_\varepsilon = \varepsilon_3 \cdot S'$. Thus, by Lemma 2.10, $K_\varepsilon = K'$ and $S_\varepsilon = S'$.

Corollary 2.12. *There is a unique expansion of the combinatory algebra $\langle P\omega, \cdot \rangle$ to a stable combinatory model $\langle P\omega, \cdot, K, S, \varepsilon \rangle$ which is also KS-stable.*

Proof. The proof follows from Theorems 2.1 and 2.11.

(See Barendregt [3, Section 5] for other results on unique expansions of combinatory algebras.)

Finally we note that although a change in the coding of ordered pairs or finite sets will change the functional behavior of individual elements, and in particular result in a new ε, the results in this paper show that ε is unique for each coding scheme.

3. Interiors of combinatory models

Now we turn briefly from $P\omega$ to more general models. Let $\mathscr{D} = \langle D, \cdot, \varepsilon \rangle$ be a combinatory model. We say $\mathscr{D}$ is extensional if $\mathscr{D}$ satisfies:

(∗) for all $d_0, d_1 \in D$, if $\forall d \in D\,(d_0 \cdot d = d_1 \cdot d)$, then $d_0 = d_1$.

Since we can interpret λ-terms in combinatory models, one can show that extensionality is equivalent to $\mathscr{D}$ satisfying the following axiom:

(η) $\lambda y.(uy) = u$ for y not free in u.

Let $\mathscr{D} = \langle D, \cdot, \varepsilon \rangle$ be any combinatory model. The interior of $\mathscr{D}$, $\mathscr{D}^0$, is the subalgebra of $\mathscr{D}$ consisting of the interpretations of all closed λ-terms (a term is closed if it has no free variables). Alternatively, we can define $\mathscr{D}^0$ to be the subalgebra of $\mathscr{D}$ generated by $K = K_\varepsilon = \|\lambda xy.x\|\sigma$ and $S = S_\varepsilon = \|\lambda xyz.xz(yz)\|\sigma$.

Proposition 3.1. *Let $\mathscr{D} = \langle D, \cdot, \varepsilon \rangle$ be a non-extensional combinatory model such that $\mathscr{D}^0 \models \eta$. Then $\langle D^0, \cdot \rangle$ has more than one SK-expansion.*

Proof. $K_\varepsilon = \|\lambda xy.x\|\sigma$ and $S_\varepsilon = \|\lambda xyz.(yz)\|\sigma$ certainly satisfy the (k) and (s) axioms and, as interpretations of closed terms, are members of D^0. Let $K' = \|\lambda xyz.xz\|\sigma$ and $S' = \|\lambda xyzw.xz(yz)w\|\sigma$. Note that $K', S' \in D^0$. If $K' = K_\varepsilon$, then, for all $d, e \in D$,

$$\begin{aligned}\|\lambda z.xz\|\sigma &= [(\lambda xyz.xz)xy\|\sigma = K' \cdot \sigma(x) \cdot \sigma(y) \\ &= K \cdot \sigma(x) \cdot \sigma(y) = \sigma(x) = \|x\|\sigma,\end{aligned}$$

and thus $\mathscr{D} \models \eta$. Since, however, $\mathscr{D}$ is non-extensional, $K' \neq K_\varepsilon$. Nevertheless since $\mathscr{D}^0 \models \eta$,

$$K' \cdot d \cdot e = \|\lambda z.xz\|\sigma\{d/x\} = \|x\|\sigma\{d/x\} = d.$$

A similar argument works for S'.

Lambda algebras may be characterized as the set of all substructures or homomorphic images of KS-expansions of combinatory models (see [2] for the actual definition of lambda algebras). These algebras are important since (1) all equations between closed terms provable in λ-calculus are true in lambda algebras, and (2) there exist mechanical procedures for building combinatory models from lambda algebras. All lambda algebras are combinatory algebras but need not be combinatory models. In Proposition 3.1, $\langle D^0, \cdot, K_\varepsilon, S_\varepsilon\rangle$ is in fact a lambda algebra since it is a substructure of $\langle D, \cdot, K_\varepsilon, S_\varepsilon\rangle$. However, $\langle D^0, \cdot, K', S'\rangle$ is not a lambda algebra (nor is any combination of $K_\varepsilon, S_\varepsilon, K', S$ aside from the original $K_\varepsilon, S_\varepsilon$). The following conjecture (suggested by Meyer) is still open.

Conjecture. *Let $\langle D, \cdot, \varepsilon\rangle$ be a combinatory model. Then there exist unique K and S such that $\langle D^0, \cdot, K, S\rangle$ is a lambda algebra.*

Appendix A. Proofs of Lemmas 2.2, 2.3 and 2.4

Before giving the proofs of the above-mentioned lemmas we note the following useful technical result.

Lemma A.1. *If $e_m \in P\omega$ is a singleton and ε' satisfies $(\varepsilon 1)$, then $\varepsilon' \cdot e_m = \{(n, p) \mid (m, n, p) \in \varepsilon'\}$.*

Proof. Let null be the index of the empty set, i.e., $e_{\text{null}} = 0$ (thus null $= 0$ in the standard encoding of sets). By definition,

$$\varepsilon' \cdot e_m = \{(n, p) \mid \exists e_k \subseteq e_m.\ (k, n, p) \in \varepsilon'\}.$$

Since e_m is a singleton, its only subsets are e_m itself and $0 = e_{\text{null}}$. Thus

$$\varepsilon' \cdot e_m = \{(n, p) \mid (m, n, p) \in \varepsilon'\} \cup \{(n, p) \mid (\text{null}, n, p) \in \varepsilon'\}.$$

Suppose $(\text{null}, n, p) \in \varepsilon'$. Thus $p \in \varepsilon' \cdot e_{\text{null}} \cdot e_n$. However, $e_{\text{null}} \cdot e_n = 0 \cdot e_n = 0$, contradicting $(\varepsilon 1)$. Thus $\{(n, p) \mid (\text{null}, n, p) \in \varepsilon'\} = 0$ and we are done.

We are now ready to proceed with the proofs of the lemmas.

Proof of Lemma 2.2. First suppose $\varepsilon' \not\supseteq \varepsilon_0$. We will show that this implies that $(\varepsilon 1)$ fails. Since $\varepsilon' \not\supseteq \varepsilon_0$, there is an $(m_0, n_0, p_0) \in \varepsilon_0 - \varepsilon'$. By definition of ε_0, $e_{m_0} = \{(n_0, p_0)\}$.

Since e_{m_0} is a singleton, by Lemma A.1,

$$\varepsilon' \cdot e_{m_0} \cdot e_{n_0} = \{p \mid \exists e_q \subseteq e_{n_0}, (m_0, q, p) \in \varepsilon'\},$$

whereas $e_{m_0} \cdot e_{n_0} = \{p_0\}$. Since $(m_0, n_0, p_0) \notin \varepsilon'$ by assumption, we must have $e_q \subsetneq e_{n_0}$. Trivially, $p_0 \in \varepsilon' \cdot e_{m_0} \cdot e_q$. However, it is easy to see that $e_{m_0} \cdot e_q = \emptyset$ (use the fact that $e_{m_0} = \{(n_0, p_0)\}$ and $e_{n_0} \not\subseteq e_q$). Thus $\varepsilon' \cdot e_{m_0} \cdot e_q \neq e_{m_0} \cdot e_q$, violating $(\varepsilon 1)$.

For the other containment, suppose $\varepsilon' \not\subseteq \varepsilon$. Again we show that $(\varepsilon 1)$ must fail. Suppose $(m_0, n_0, p_0) \in \varepsilon' - \varepsilon$ (note that any $k \in \omega$ may be interpreted as an ordered pair, and hence as a triple, by our coding scheme). Hence by the definition of ε there is no $e_q \subseteq e_{n_0}$ such that $(q, p_0) \in e_{m_0}$, and therefore $p_0 \notin e_{m_0} \cdot e_{n_0}$. On the other hand $p_0 \in \varepsilon' \cdot e_{m_0} \cdot e_{n_0}$, so $(\varepsilon 1)$ again fails.

Proof of Lemma 2.3. Suppose $\varepsilon_1 \not\subseteq \varepsilon'$. We show that $(\varepsilon 2)$ fails. Suppose $(m_0, n_0, p_0) \in \varepsilon_1 - \varepsilon'$, and hence, by the definition of ε_1, $e_{m_0} = \{(q_0, p_0)\}$ for some $e_{q_0} \subseteq e_{n_0}$. By Lemma 2.2, $\varepsilon_0 \subseteq \varepsilon'$, and by the definition of ε_0, $(m_0, q_0, p_0) \in \varepsilon_0 \subseteq \varepsilon'$. Thus $q_0 \neq n_0$ and hence $e_{q_0} \subsetneq e_{n_0}$. Since e_{n_0} is a singleton, by Lemma A.1 we have $\varepsilon' \cdot e_{n_0} = \{(n, p) \mid (m_0, n, p) \in \varepsilon'\}$. Thus $(m_0, p_0) \in \varepsilon' \cdot e_{m_0}$. If $e = \{(r, p_1) \mid e_{q_0} \subseteq e_r\}$, then it is easy to verify that (1) $e_{n_0} \subseteq e$, (2) $(n_0, p_0) \in e$, and (3) for all $d \in P\omega$, $e \cdot d = e_{m_0} \cdot d$. However, we can show that $\varepsilon' \cdot e \neq \varepsilon' \cdot e_{n_0}$, violating $(\varepsilon 2)$. It is enough to show $(n_0, p_0) \in \varepsilon' \cdot e$ since we know that $(n_0, p_0) \notin \varepsilon' \cdot e_n$. Let $e_r = \{(m_0, p_0)\} \subseteq e$ (by (2) above). By the definition of ε_0, we have $(r, n_0, p_0) \in \varepsilon_0 \subseteq \varepsilon'$. Thus $(n_0, p_0) \in \varepsilon' \cdot e$.

Proof of Lemma 2.4

$$\begin{aligned}
\varepsilon_1 \cdot \varepsilon_1 &= \{(m, n, p) \mid \exists e_q \subseteq \varepsilon_1, (q, m, n, p) \in \varepsilon_1\} \\
&= \{(m, n, p) \mid \exists e_q \subseteq \varepsilon_1 \ \exists e_r \subseteq e_m, e_q = \{(r, n, p)\}\} \\
&= \{(m, n, p) \mid \exists e_r \subseteq e_m, (r, n, p) \in \varepsilon_1\} \\
&= \{(m, n, p) \mid \exists e_r \subseteq e_m \ \exists e_q \subseteq e_n, e_r = \{(q, p)\}\} \\
&= \{(m, n, p) \mid \exists e_q \subseteq e_n, (q, p) \in e_m\} \\
&= \varepsilon.
\end{aligned}$$

Thus $\varepsilon_1 \cdot \varepsilon_1 = \varepsilon$. By the monotonicity of application and since $\varepsilon_1 \subseteq \varepsilon' \subseteq \varepsilon$ by Lemmas 2.2 and 2.3, $\varepsilon = \varepsilon_1 \cdot \varepsilon_1 \subseteq \varepsilon' \cdot \varepsilon' \subseteq \varepsilon \cdot \varepsilon = \varepsilon$. Thus all $\subseteq$'s become $=$'s and $\varepsilon' \cdot \varepsilon' = \varepsilon$.

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